Pergamon

PII: S0021–8928(97)00032-4

J. Appl. Maths Mechs, Vol. 61, No. 2, pp. 253–263, 1997

0021–8928/97 $24.00+0.00

# A METHOD OF SOLVING PLANE INITIAL AND BOUNDARY-VALUE PROBLEMS OF THE DYNAMIC THEORY OF ELASTICITY†

S. V. SHMEGERA

Irpen'

(*Received* 4 *July* 1994)

A method for solving plane initial and boundary-value problems (IBVP) in the dynamic theory of elasticity (DTE), in which the solutions are represented in the form of a continuous planar superposition of arbitrary analytic functions which are complex plane waves of arbitrary shape is described. By representing the solutions in this form it is possible to use the Radon transformation, by means of which plane IBVP of the DTE reduce to boundary-value problems of the theory of functions of a complex variable: in the simplest cases of those of Dirichlet or Keldysh–Sedov, and in more complicated cases to Riemann–Hilbert or Riemann (matching) problems. The method is illustrated on some basic IBVP of the DTE in a half-plane. An analytic solution for the basic mixed problem of the DTE, in which there is no boundary condition that applies over the entire infinite interval of the half-plane boundary, is found for the first time. 

In this paper we consider initial and boundary-value problems of the dynamic theory of elasticity for a half-plane or those which can be reduced to such in mathematical terms. All the known analytic solutions have been obtained on problems (see [1], for example) in which at least one of the two boundary conditions is satisfied over the entire infinite interval of the half-plane boundary. In that case, existing methods, such as the Smirnov–Sobolev method (for self-similar problems) or using integrals transforms based on Fourier and (or) Laplace integrals with the Wiener–Hopf technique, can be used. In the more complicated case where there is no boundary condition that applies over the entire infinite interval of the half-plane boundary (classified in [2] as the basic mixed problem) the considerable difficulties that arise when the known methods are used are impossible to overcome within the given framework.

The method described here can be used for the analytic solution of problems of this kind. It involves representing the solutions of the wave equations, to a set of which, as we know (see [1, 3], for example) the equations of the DTE can be reduced, as a continuous (integral) superposition of arbitrary functions with a unified complex-valued dependence on the coordinates and time. For a fixed value of a parameter, these analytic functions of a complex variable are complex plane waves of arbitrary form. In that case the kinematic and dynamic characteristics of motion of a homogeneous and isotropic elastic medium can be represented in the form of integrals of two arbitrary analytic functions, and the boundary values (on the boundary of the half-plane) of those representations turn out to be connected with the inversion formulae of the two-dimensional Radon transformation. By this means the IBVP of the DTE can be reduced to simple systems of Riemann–Hilbert boundary-value problems for two functions, which are easy to solve.

To demonstrate how the method works, we will give examples of the analytic solution of basic IBVP of the DTE for a half-plane: a problem with boundary conditions of just one type, a mixed problem, and the basic mixed problem. Thorough studies have been made of the first two problems by existing methods, and they are presented for illustrative purposes, both to explain some details of the use of the Radon transformation and to compare them with the new method. The basic mixed problem is considered in more detail, with the properties of the general solution illustrated by the example of the transient problem of a punch which is rigidly coupled to an elastic half-plane.

## 1. FORMALISM OF THE DTE. THE REPRESENTATION OF SOLUTIONS

Suppose that a homogeneous isotropic body with shear modulus $\mu$ and velocities of propagation of longitudinal and transverse elastic waves $c_1$ and $c_2$, respectively, is in plane deformation with zero volume forces. Then the expressions for the components of the displacement vector $\mathbf{w} = \{u, v\}$ and non-zero components of the stress tensor can be written in the form

†*Prikl. Mat. Mekh.* Vol. 61, No. 2, pp. 263–274, 1997.

$$u = \varphi_{1,x} + \varphi_{2,y},\ v = \varphi_{1,y} - \varphi_{2,x},\ \sigma_x = \mu\left[c_2^{-2}\ddot{\varphi}_1 - 2(\varphi_{1,yy} - \varphi_{2,xy})\right]$$
$$\sigma_y = \mu\left[c_2^{-2}\ddot{\varphi}_1 - 2(\varphi_{1,xx} + \varphi_{2,xy})\right] \tag{1.1}$$
$$\tau_{xy} = \mu\left[c_2^{-2}\ddot{\varphi}_2 - 2(\varphi_{2,xx} - \varphi_{1,xy})\right]$$
$$(\dot{\ }) \equiv \partial/\partial t,\ (\ )_{,p} \equiv \partial/\partial p\ (p = x,\ y)$$

The potentials $\varphi_j$ satisfy the wave equations

$$\varphi_{j,xx} + \varphi_{j,yy} = c_j^{-2}\ddot{\varphi}_j \tag{1.2}$$

(here and everywhere below $j = 1, 2$).

We shall represent the solutions of Eqs (1.2) as an integral superposition of arbitrary plane waves

$$\varphi_j(x,\ y,\ t) = \frac{1}{2\pi i}\int_\Gamma F_j(z_j(x,\ y,\ t,\ c),\ c)dc \tag{1.3}$$

Here $F_j(z_j)$ are arbitrary twice-differentiable or (if $z_j$ are complex) analytic functions which are plane waves of arbitrary form for fixed values of the variable $c$, $\Gamma$ is an arbitrary contour in the complex plane of $c$ and the functions $z_j$ have the form

$$z_j = k_j(c)t + m_j(c)x + n_j(c)y \tag{1.4}$$

Substituting representations (1.3), allowing for (1.4), into Eqs (1.2), since $F_j(z_j)$ is arbitrary, we obtain the necessary and sufficient condition for expressions (1.3) to be solutions of wave equations (1.2)

$$k_j^2 = c_j^2(m_j^2 + n_j^2) \tag{1.5}$$

It is clear from the form of expressions (1.4) and (1.5) that the arbitrary functions $F_j(z_j)$ are the class of functionally invariant solutions of Eqs (1.2).

We will now consider a specific case of expressions (1.4). Taking $k_j = -c$, $m = 1$ in (1.4), from (1.5) we find $n_j = \pm i(1 - c^2/c_j^2)^{1/2}$. To pick out the single-valued branches of the radicals, we make cuts $[-\infty, -c_j]$ and $[c_j, \infty]$ along the real axis in the complex plane of $c$ and fix the branches by the conditions $(1 - c^2/c_j^2)^{1/2} > 0$ for $c = ia (a > 0)$. Then

$$z_j = \xi + i\eta_j;\ \xi = x - ct,\ \eta_j = \gamma_j y,\ \gamma_j = (1 - c^2/c_j^2)^{1/2} \tag{1.6}$$

In the case considered here, under the condition that $c < c_j$, $F_j(z_j)$ will be analytic functions of the complex variables $z_j = \xi + i\eta$.

In physical terms, the functions $F_j(z_j)$ no longer correspond to plane waves when $c < c_j$. Following Smirnov and Sobolev, we shall call them complex plane waves.

In this case, taking (1.6) into account, we will choose the solutions of Eqs (1.2) as the real parts of the representations (1.3)

$$\varphi_j(x,\ y,\ t) = \operatorname{Re}\frac{1}{2\pi i}\int_\Gamma F_j(x - ct + i\gamma_j(c)y,\ c)dc \tag{1.7}$$

The contour $\Gamma$ in the $c$ plane lies in the quadrant $\operatorname{Re} c < 0$, $\operatorname{Im} c < 0$ for $\operatorname{Re} c < 0$, and in quadrant $\operatorname{Re} c > 0$, $\operatorname{Im} c > 0$ for $\operatorname{Re} c > 0$.

Substituting (1.7) into (1.1), we obtain the following representations for the components of the displacement and stresses

$$u = \operatorname{Re}\frac{1}{2\pi i}\int_\Gamma [F_1' + i\gamma_2(c)F_2']dc,\ v = \operatorname{Re}\frac{1}{2\pi i}\int_\Gamma [i\gamma_1(c)F_1' - F_2']dc$$

$$\begin{aligned}
\sigma_x &= \mu\,\mathrm{Re}\frac{1}{2\pi i}\int_\Gamma [\omega(c)F_1'' + 2i\gamma_2(c)F_2'']dc \\
\sigma_y &= \mu\,\mathrm{Re}\frac{-1}{2\pi i}\int_\Gamma [\gamma(c)F_1'' + 2i\gamma_2(c)F_2'']dc \\
\tau_{xy} &= \mu\,\mathrm{Re}\frac{1}{2\pi i}\int_\Gamma [2i\gamma_1(c)F_1'' - \gamma(c)F_2'']dc \\
\gamma &= 1+\gamma_2^2, \quad \omega = 1-\gamma_2^2+2\gamma_1^2 \\
F_j &= F_j(z_j(c),\ c) \quad (F_j' = \partial F_j / \partial z_j,\ \ F_j'' = \partial^2 F_j / \partial z_j^2)
\end{aligned} \tag{1.8}$$

Thus, in principle, the solution of plane IBVP of the DTE can be reduced to finding two analytic functions and (or) their derivatives, that is, the boundary-value problems of the theory of functions of a complex variable, for which there are complete solution methods [4, 5]. As we shall show, this reduction can in fact be carried out using the Radon transformation.

## 2. USE OF THE RADON TRANSFORMATION. A PROBLEM WITH BOUNDARY CONDITIONS OF ONE KIND

We will first consider the problem with boundary conditions of just one kind, the first or second boundary-value problem. The first is more interesting for applications. The second can be analysed in the same way.

On the boundary of the elastic half-plane $y < 0$ let the normal and shear stresses be given

$$\sigma_y(x,\ 0,\ t) = \sigma_0(x,\ t), \quad \tau_{xy}(x,\ 0,\ t) = \tau_0(x,\ t) \quad (t > 0) \tag{2.1}$$

Zero initial conditions are specified

$$\mathbf{w} = \partial\mathbf{w} / \partial t = 0 \quad (t < 0) \tag{2.2}$$

Substituting the representations for the stresses from (1.8) into conditions (2.1), we can write the latter in the form

$$\begin{aligned}
\mathrm{Re}\frac{1}{2\pi i}\int_\Gamma [\gamma(c)F_1''^{+} + 2i\gamma_2(c)F_2''^{+}]dc &= -\mu^{-1}\sigma_0(x,\ t) \\
\mathrm{Re}\frac{1}{2\pi i}\int_\Gamma [2i\gamma_1(c)F_1''^{+} - \gamma(c)F_2''^{+}]dc &= \mu^{-1}\tau_0(x,\ t)
\end{aligned} \tag{2.3}$$

Here $F_j''^{+} = F_j''^{+}(\xi(x, t, c), c)$, with $z_j = \xi + i0$, are the boundary values of functions $F_j''^{+}(z_j)$, analytic in the half-planes $\mathrm{Im}\, z_j > 0$.

Using the relation $f^{-}(\xi) = \overline{f^{+}}(\xi)$, where the bar denotes the complex conjugate, we can rewrite the first equation of (2.3), for example, in the form

$$\frac{1}{4\pi i}\int_\Gamma \{\gamma(c)[F_1''^{+} + F_1''^{-}] + 2i\gamma_2(c)[F_2''^{+} - F_2''^{-}]\}dc = -\mu^{-1}\sigma_0(x,\ t) \tag{2.4}$$

where $F_j''^{-} = F_j''^{-}(\xi(x, t, c), c)$ are the boundary values, for $z_j = \xi - i0$, of the functions $F_j''^{-}(z_j)$, which are analytic in half-planes $\mathrm{Im}\, z_j < 0$.

Using the Sokhotskii–Plemelj formulae [6], from (2.4) we obtain

$$-\frac{1}{4\pi^2}\int_\Gamma \int_{-\infty}^{\infty} \frac{\Sigma(\xi',\ c)}{\xi' - \xi(x,\ t,\ c)} d\xi' dc = \sigma_0(x,\ t) \tag{2.5}$$

Here $\Sigma(\xi)$ is a function which satisfies the Hölder condition [5], including the point at infinity.

Expression (2.5) is the inversion formula for the two-dimensional Radon transformation (RT) (see [7, 8] for example) if $\Sigma(\xi, c)$ is the RT of the function $\sigma_0(x, t)$

$$\Sigma(\xi,\ c)=\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma_0(x,\ t)\delta(x-ct-\xi)dxdt \tag{2.6}$$

where $\delta(\cdot)$ is the Dirac delta-function. Hence expressions (2.3) or (2.4), which implicitly contain the boundary values of Cauchy-type integrals, are analogues of the classical inversion formulae for the two-dimensional RT.

It will be more convenient to use a formal separation of the operations in expressions (2.5) and (2.6) which differs slightly from that used in [7, 8], where it is strictly the RT that is referred to as the integral part of formula (2.6), and the differentiation operation appears in the inversion formula (2.5).

Applying the RT in form (2.6) to system (2.3) and allowing for the fact that transformation of the left-hand sides of expressions (2.3) actually eliminates the operation of integration with respect to variable $c$ (this follows from (2.5) and the Sokhotskii–Plemelj formulae), we obtain

$$\begin{aligned}&\operatorname{Re}[\gamma F_1''^{+}(\xi)+2i\gamma_2 F_2''^{+}(\xi)]=-\mu^{-1}\Sigma(\xi)\\&\operatorname{Re}[2i\gamma_1 F_1''^{+}(\xi)-\gamma F_2''^{+}(\xi)]=\mu^{-1}\mathrm{T}(\xi)\end{aligned} \tag{2.7}$$

where $\mathrm{T}(\xi)$ is defined by an expression similar to (2.6), with $\sigma_0(x, t)$ replaced by $\tau_0(x, t)$.

Regarding the equations of system (2.7) as independent Dirichlet problems for the expressions in square brackets, using the Schwarz integral [6] for $\operatorname{Im} z_j = 0$ we find

$$F_j''^{+}(\xi)=\frac{1}{\mu R}\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty}\Lambda_j(\xi')\frac{d\xi'}{\xi'-\xi}+\Lambda_j(\xi)\right\} \tag{2.8}$$

$$\Lambda_1(\xi)=-\gamma\Sigma(\xi)+2i\gamma_2\mathrm{T}(\xi),\quad \Lambda_2(\xi)=-2i\gamma_1\Sigma(\xi)-\gamma\mathrm{T}(\xi)$$

$$R=\gamma^2-4\gamma_1\gamma_2$$

Then from (2.8), using the properties of a Cauchy integral [6], we obtain

$$F_j''^{+}(z_j)=\frac{1}{\mu R\pi i}\int_{-\infty}^{\infty}\Lambda_j(\xi)\frac{d\xi}{\xi-z_j} \tag{2.9}$$

Integrating expressions (2.9) with respect to the variables $z_j$ we find

$$F_j'^{+}(z_j)=\frac{1}{\mu R\pi i}\int_{-\infty}^{\infty}\Lambda_j(\xi)\ln|\xi-z_j|d\xi \tag{2.10}$$

If the functions $F_j''^{+}(z_j)$ and $F_j''^{+}(z_j)$ are known, all the required kinematic and dynamic characteristics of motion of an elastic medium in a half-plane can be found using representations (1.8).

## 3. MIXED BOUNDARY CONDITIONS

We will now consider the problem in which one of the two boundary conditions is mixed: the condition on one part of the $y = 0$ is different from that on the other, while the second condition applies over the whole axis $y = 0$. Then the points at which the type of boundary condition changes might move along the boundary of the half-plane with arbitrary variable velocity. The method can be described by considering a problem with the following boundary and initial conditions

$$\begin{aligned}&\sigma_y(x,\ 0,\ t)=\sigma_0(x,\ t)\ \ (x<l(t),\ t>0)\\&\tau_{xy}(x,\ 0,\ t)=\tau_0(x,\ t)\ \ (-\infty<x<\infty,\ t>0)\\&v(x,\ 0,\ t)=v_0(x,\ t)\ \ (x\geqslant l(t),\ t>0)\\&\mathrm{w}=\partial\mathrm{w}/\partial t=0\ \ (t<0)\end{aligned} \tag{3.1}$$

Here $l(t)$ is an arbitrary function of time.

Many problems in fracture dynamics [9, 10] involving the penetration and (or) motion of smooth punches [11, 12], and of the diffraction of elastic waves [13], among others, reduce to problems with conditions (3.1).

For a unique solution, the displacement vector must be bounded and continuous in the neighbourhood of the point where the boundary condition changes type, allowing for an integrable singularity in the stresses and displacement velocity at that point [1]

$$\mathbf{w}(x, y, t) = \mathbf{a}(t) + O(r^{\varepsilon}), \quad r \to 0 \quad (r = [(x - l(t))^2 + y^2]^{1/2}) \tag{3.2}$$

If $dl/dt = 0$, we need merely put $\varepsilon > 0$ in (3.2). But if $dl/dt \neq 0$, we must use the stronger condition: $\varepsilon \geqslant 1/2$ which, as we know [1], allows for possible absorption or emission of energy at the point where the boundary condition changes type.

Substituting the representations for the stresses and displacements in (1.8) into the boundary conditions in (3.1), applying the RT in the form (2.6) to the result (taking the initial conditions of (3.1) into account) and putting

$$\begin{Bmatrix} \Sigma(\xi) \\ \mathrm{T}(\xi) \\ V(\xi) \end{Bmatrix} = \frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{Bmatrix} \sigma_0(x, t) H(l(t) - x) \\ \tau_0(x, t) \\ \upsilon_0(x, t) H(x - l(t)) \end{Bmatrix} H(t) \delta(x - ct - \xi) dx dt \tag{3.3}$$

(the Heaviside functions $H(\bullet)$ have been introduced to emphasize that the carriers of the functions $\sigma_0(x, t)$, $\tau_0(x, t)$ and $u_0(x, t)$ have limiting values), we obtain a system of Riemann–Hilbert boundary-value problems for the functions $F'_j(z_j)$ and $F''_j(z_j)$, which are analytic in the half-planes $\operatorname{Im} z_j > 0$

$$\operatorname{Re}[\gamma F_1''(\xi) + 2i\gamma_2 F_2''(\xi)] = \begin{cases} -\mu^{-1}\Sigma(\xi) & (\xi < l_* - ct) \\ 0 & (\xi > l_*) \end{cases} \tag{3.4}$$

$$\operatorname{Re}[2i\gamma_1 F_1''(\xi) - \gamma F_2''(\xi)] = \begin{cases} \mu^{-1}\mathrm{T}(\xi) & (\xi < 0) \\ 0 & (\xi > 0) \end{cases} \tag{3.5}$$

$$\operatorname{Re}[i\gamma_1 F_1'(\xi) - F_2'(\xi)] = V(\xi) \quad (l_* - ct < \xi < l_*) \tag{3.6}$$

Here $l_* = l(t_*)$, where $t_*$ is a root of the equation $\xi + ct_* - l(t_*) = 0$.

Note that the intervals on the $\operatorname{Im} z_j = 0$ axis on which conditions (3.4)–(3.6) are defined follow at once from (3.3) and the theorem on the properties of a carrier of the RT [7, 8].

Condition (3.5) can be regarded as a Dirichlet problem for the expression in square brackets, which can be solved using the Schwarz integral [6] and, when $\operatorname{Im} z_j = 0$, has the form

$$2i\gamma_1 F_1''(\xi) - \gamma F_2''(\xi) = \mu^{-1}\left[\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\mathrm{T}(\xi')}{\xi' - \xi} d\xi' + \mathrm{T}(\xi)\right] = \mu^{-1}\Omega(\xi) \tag{3.7}$$

Differentiating Eq. (3.6) with respect to $\xi$ and using (3.7) to eliminate in turn the functions $F'_1(\xi)$ and $F''_2(\xi)$ for the conditions (3.4) and (3.6) (after differentiating the latter), we obtain two separate independent Riemann–Hilbert problems for $F'_1(z_1)$ and $F''_2(z_2)$. It is better to use the known relation between Riemann–Hilbert and Riemann (matching) problems [4, 5] and rewrite these in the form of Riemann problems

$$F_1''^{+}(\xi) - \frac{\bar{R}}{R} F_1''^{-}(\xi) = \begin{cases} \dfrac{2}{\mu R}[2 \operatorname{Re} i\gamma_2 \Omega(\xi) - \gamma\Sigma(\xi)] & (\xi < l_* - ct) \\ 0 & (\xi > l_*) \end{cases}$$

$$F_1''^{+}(\xi) + F_1''^{-}(\xi) = \frac{2}{i\gamma_1(\gamma_2^2 - 1)}[\gamma V'(\xi) - \mu^{-1} \operatorname{Re}\Omega(\xi)] \quad (l_* - ct < \xi < l_*) \tag{3.8}$$

$$F_2''^{+}(\xi)+\frac{\bar{R}}{R}F_2''^{-}(\xi)=\begin{cases}\dfrac{i\gamma_1}{\mu R}\left[\operatorname{Re}\dfrac{\gamma}{i\gamma_1}\Omega(\xi)-2\Sigma(\xi)\right] & (\xi<l_*-ct)\\ 0 & (\xi>l_*)\end{cases}$$

$$F_2''^{+}(\xi)-F_2''^{-}(\xi)=\frac{2}{\gamma_2^2-1}[2V'(\xi)-\mu^{-1}\Omega(\xi)](l_*-ct<\xi<l_*) \tag{3.9}$$

Here $V' = dV/\partial\xi$, the function $V'$ is defined in (2.8) and $\bar{R}$ is the complex conjugate of $R$.

Problems (3.8) and (3.9) are referred to as Riemann problems with discontinuous coefficients [4, 5]. In the class of functions which vanish at infinity, provided that the functions $\Sigma(\xi)$, $\Omega(\xi)$ and $R(\xi)$ satisfy a Hölder condition (excluding any points of discontinuity of the coefficients), known formulae [4, 5] can be used to write their solutions in the form

$$F_1''(z_1)=\frac{1}{\pi i}G_1(z_1)\left\{\frac{1}{\mu R}\int_{-\infty}^{l_*-ct}G_1^{-1}(\xi)[2\operatorname{Re}i\gamma_2\Omega(\xi)-\gamma\Sigma(\xi)]\frac{d\xi}{\xi-z_1}+\right.$$
$$\left.+\frac{1}{i\gamma_1(\gamma_2^2-1)}\int_{l_*-ct}^{l_*}G_1^{-1}(\xi)[\gamma V'(\xi)-\mu^{-1}\operatorname{Re}\Omega(\xi)]\frac{d\xi}{\xi-z_1}\right\} \tag{3.10}$$

$$F_2''(z_2)=\frac{1}{\pi i}G_2(z_2)\left\{\frac{i\gamma_1}{\mu R}\int_{-\infty}^{l_*-ct}G_2^{-1}(\xi)[\operatorname{Re}\frac{\gamma}{2i\gamma_1}\Omega(\xi)-\Sigma(\xi)]\frac{d\xi}{\xi-z_2}+\right.$$
$$\left.+\frac{1}{\gamma_2^2-1)}\int_{l_*-ct}^{l_*}G_2^{-1}(\xi)[2V'(\xi)-\mu^{-1}\operatorname{Re}\Omega(\xi)]\frac{d\xi}{\xi-z_2}\right\}$$

Here

$$G_j(z_j)=\left(\frac{l_*-z_j}{l_*-ct-z_j}\right)^{-\alpha+1/2},\quad \alpha=\frac{1}{2\pi i}\ln\frac{\bar{R}}{R}$$

provided that $0\leqslant\arg(\bar{R}/R)<2\pi$, which ensures that the solution of problems (3.8) and (3.9) is unique and also that the solution of the original problem (3.1) behaves in accordance with conditions (3.2) at the point where the type of boundary condition changes.

Now, knowing the functions $F_j''(z_1)$, we can use (1.8) to obtain expressions for the stresses in the half-plane. The displacements can be found from the stresses or by integrating expressions (3.10) with respect to $z_j$ to find $F_j'(z_j)$, and then using the representations for the displacements from (1.8).

## 4. THE BASIC MIXED PROBLEM

A characteristic feature of the basic mixed problem is that no boundary condition applies over the entire half-plane boundary. As we have already noted, existing methods cannot be used to obtain an analytic solution of this problem (cf. [1, p. 244]). The Smirnov–Sobolev method yields a complicated system of Riemann–Hilbert boundary-value problems with variable coefficients or the equivalent system of integral equations, for which there are no effective methods of solution. The method of Fourier and Laplace integral transformations requires the matrices of the functions to be factorized, which is impossible to do in practice.

With the method described here, the basic mixed problem reduces to a comparatively simple style of Riemann–Hilbert problems with constant coefficients, which is easy to solve.

Suppose that for the elastic half-plane $y<0$ the following boundary and initial conditions are specified

$$\begin{aligned}&\sigma_y(x,0,t)=\sigma_0(x,t),\ \tau_{xy}(x,0,t)=\tau_0(x,t)\ (x<0,\ t>0)\\&u(x,0,t)=u_0(x,t),\ v(x,0,t)=v_0(x,t)\ (x\geqslant 0,\ t>0)\\&\mathbf{w}=\partial\mathbf{w}/\partial t=0\ (t<0)\end{aligned} \tag{4.1}$$

We ensure a unique solution by imposing a constraint on the behaviour of the displacement vector in the neighbourhood of the point where the boundary condition changes type

$$\mathbf{w}(x,\ y,\ t)=\mathbf{a}(t)+O(r^{\varepsilon}),\ \ \varepsilon>0,\ \ r\to 0\ \ (r=(x^2+y^2)^{1/2} \tag{4.2}$$

Substituting the representations of the displacements and stresses of (1.8) into the boundary conditions of (4.1) and applying the RT in the form (2.6) to the resulting relations, we obtain a system of boundary-value problems for functions $F_j'(z_j)$ and $F'_j{}'(z_j)$, which are analytic in the half-planes $\operatorname{Im} z_j > 0$

$$\operatorname{Re}[\gamma F_1''(\xi)+2i\gamma_2 F_2''(\xi)]=\begin{cases}-\mu^{-1}\Sigma_0(\xi)\ (\xi<-ct)\\ 0(\xi>0)\end{cases} \tag{4.3}$$

$$\operatorname{Re}[2i\gamma_1 F_1''(\xi)-\gamma F_2''(\xi)]=\begin{cases}\mu^{-1}\mathrm{T}_0(\xi)\ (\xi<-ct)\\ 0(\xi>0)\end{cases} \tag{4.4}$$

$$\operatorname{Re}[F_1'(\xi)+i\gamma_2 F_2'(\xi)]=U_0(\xi)\ \ (-ct<\xi<0) \tag{4.5}$$

$$\operatorname{Re}[i\gamma_1 F_1'(\xi)-F_2'(\xi)]=V_0(\xi)\ \ (-ct<\xi<0) \tag{4.6}$$

$$\begin{Bmatrix}\Sigma_0(\xi)\\ \mathrm{T}_0(\xi)\\ U_0(\xi)\\ V_0(\xi)\end{Bmatrix}=\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{Bmatrix}\sigma_0(x,\ t)H(-x)\\ \tau_0(x,\ t)H(-x)\\ u_0(x,\ t)H(x)\\ v_0(x,\ t)H(x)\end{Bmatrix}H(t)\delta(x-ct-\xi)dxdt$$

As in the previous case (Section 3), system (4.3)–(4.6) can be solved in terms of the functions $F_j(z_j)$. Here, however, it is better to use a method which gives the solution in the form of expressions which are the RT of the stresses on the half-plane boundary. This is done using formulae (2.8), in this case taking the functions $\Sigma(\xi)$ and $T(\xi)$ on the boundary of the half-planes $\operatorname{Im} z_j > 0$ to be unknown (in fact, they are unknown the interval $-ct < \xi < 0$).

We now differentiate Eqs (4.5) and (4.6) with respect to $\xi$ and substitute (4.3)–(4.6) (after differentiating) into expressions (2.8). Thus, using the Sokhotskii–Plemelj formulae [6], we introduce the following relations

$$\Sigma(\xi)=P_1^+(\xi)-P_1^-(\xi),\ \ \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\Sigma(\xi')}{\xi'-\xi}d\xi'=P_1^+(\xi)+P_1^-(\xi) \tag{4.7}$$

$$\mathrm{T}(\xi)=P_2^+(\xi)-P_2^-(\xi),\ \ \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{T}(\xi')}{\xi'-\xi}d\xi'=P_2^+(\xi)+P_2^-(\xi) \tag{4.8}$$

where, for $z_j = \xi \pm i0$, $P_j^+(\xi)$ and $P_j^-(\xi)$ are the boundary values of certain functions $P_j^+(z_j)$ and $P_j^-(z_j)$, which are analytic in the half-planes $\operatorname{Im} z_j > 0$ and $\operatorname{Im} z_j < 0$ respectively. As a result we obtain a system of Riemann matching boundary-value problems for the functions $P_j^{\pm}(z_j)$, with the following relation between their limiting values on the $\operatorname{Im} z_j = 0$ axis

$$P_1^+(\xi)-P_1^-(\xi)=\begin{cases}\Sigma_0(\xi)\ (\xi<-ct)\\ 0(\xi>0)\end{cases},\ \ P_2^+(\xi)-P_2^-(\xi)=\begin{cases}\mathrm{T}_0(\xi)\ (\xi<-ct)\\ 0(\xi>0)\end{cases} \tag{4.9}$$

$$\frac{Q}{R}P_1^+(\xi)-\frac{\overline{Q}}{R}P_1^-(\xi)+\frac{N_2}{R}P_2^+(\xi)-\frac{\overline{N}_2}{R}P_2^-(\xi)=\mu U_0'(\xi)(-ct<\xi<0) \tag{4.10}$$

$$\frac{N_1}{R}P_1^+(\xi)-\frac{\overline{N}_1}{R}P_1^-(\xi)-\frac{Q}{R}P_2^+(\xi)+\frac{\overline{Q}}{\overline{R}}P_2^-(\xi)=\mu V_0'(\xi)(-ct<\xi<0) \tag{4.11}$$

$$Q=2\gamma_1\gamma_2-\gamma,\ \ N_j=i\gamma_j(1-\gamma_2^2) \tag{4.12}$$

$U'_0 = \partial U_0/\partial\xi$, $V_0 = \partial V_0/\partial\xi$, the function $R$ is defined in (2.8) and the bar denotes the complex conjugate.

We now multiply Eq. (4.11) by a certain constant $A$ and add it term-by-term to (4.10)

$$\frac{1}{R}(Q+AN_1)[P_1^+(\xi)+BP_2^+(\xi)]-\frac{1}{\overline{R}}(\overline{Q}+A\overline{N}_1)[P_1^-(\xi)+CP_2^-(\xi)]=\mu[U_0'(\xi)+AV_0'(\xi)] \quad (4.13)$$

$$B=\frac{N_2-AQ}{Q+AN_1},\quad C=\frac{\overline{N}_2-A\overline{Q}}{\overline{Q}+A\overline{N}_1} \quad (4.14)$$

We choose $A$ so that $B = C$. Then the condition obtained for $A$ from (4.14) leads to a quadratic equation, which has a solution of the form

$$A_n=\frac{1}{2}q_1^{-1}[p+(-1)^{n-1}(p^2-4q_1q_2)^{1/2}]\ (n=1,2) \quad (4.15)$$

$$q_j=N_j\overline{Q}-\overline{N}_jQ,\ \ p=N_1\overline{N}_2-\overline{N}_1N_2$$

If $\bar{R} \equiv R$, $\bar{Q} \equiv Q$ and $\bar{N}_j \equiv N_j$, we have $B = C$ for any $A$. In that case, as we can see from (4.13) and (4.14), for $A_1$ and $A_2$ it is better to take the values

$$A_1=\frac{N_2}{Q},\quad A_2=-\frac{Q}{N_1} \quad (4.16)$$

Substituting the values of $A_1$ and $A_2$ found using (4.15) term-by-term into (4.14) and then substituting $A_1$ and $A_2$ and the corresponding values of $B_1$ and $B_2$ (since $B = C$) into (4.13), we obtain

$$[P_1^+(\xi)+B_nP_2^+(\xi)]-\lambda_n[P_1^-(\xi)+B_nP_2^-(\xi)]=\nu_n[U_0'(\xi)+A_nV_0'(\xi)]$$
$$(n=1,2;\ -ct<\xi<0) \quad (4.17)$$

$$B_n=\frac{N_2-A_nQ}{Q+A_nN_1},\quad \lambda_n=\frac{R}{\overline{R}}\frac{\overline{Q}+A_n\overline{N}_1}{Q+A_nN_1},\quad \nu_n=\frac{\mu R}{Q+A_nN_1} \quad (4.18)$$

Conditions (4.9) can be written in the form

$$[P_1^+(\xi)+B_nP_2^+(\xi)]-[P_1^-(\xi)+B_nP_2^-(\xi)]=\begin{cases}\Sigma_0(\xi)+B_n\mathrm{T}_0(\xi)(\xi<-ct)\\ 0(\xi>0)\end{cases} \quad (4.19)$$

This method of reducing the system of Riemann–Hilbert boundary-value problems (4.3)–(4.6) to two independent Riemann problems (for $n = 1$ and $n = 2$ in (4.17)–(4.19) respectively) is taken from [14], where it was used to solve problems of static contact for an anisotropic half-plane.

The solutions of problems (4.17) and (4.19) for $z_j = \xi + i0$ which vanish at infinity can be written out for the case of discontinuous coefficients using the formulae of [4, 5]. Considering the result (for $n = 1, 2$) as a system of two algebraic equations with respect to the functions $P_1^+(\xi)$ and $P_2^+(\xi)$, we find

$$P_j^+(\xi)=\frac{1}{2\pi i}\frac{(-1)^{j-1}}{B_2-B_1}\sum_{n=1}^{2}(-1)^{n-1}\nu_nB_{3-n}^{2-j}\Bigg\{G_n(\xi)\int_{-ct}^{0}G_n^{-1}(\xi')\frac{U_0'(\xi')+A_nV_0'(\xi')}{\xi'-\xi}d\xi'+$$
$$+\pi i[U_0'(\xi)+A_nV_0'(\xi)]+\frac{1}{\mu R}G_n(\xi)\int_{-\infty}^{-ct}G_n^{-1}(\xi')\frac{\Sigma_0(\xi')+B_n\mathrm{T}_0(\xi')}{\xi'-\xi}d\xi'+$$
$$+\frac{\pi i}{\mu R}[\Sigma_0(\xi)+B_n\mathrm{T}_0(\xi)]\Bigg\} \quad (4.20)$$

$$G_n(\xi)=\left(\frac{\xi}{\xi+ct}\right)^{\alpha_n},\quad \alpha_n=-\frac{1}{2\pi i}\ln\lambda_n,\ -2\pi\leq\arg\lambda_n<0$$

In the notation of (4.7) and (4.8), we have

$$\Sigma(\xi) = 2\,\mathrm{Re}\, P_1^+(\xi), \quad \mathrm{T}(\xi) = 2\,\mathrm{Re}\, P_2^+(\xi) \tag{4.21}$$

Knowing the functions $\Sigma(\xi)$ and $\mathrm{T}(\xi)$, and now also knowing their values on the entire boundary of the half-plane $\mathrm{Im}\, z_j > 0$, we can find $F'_j(z_j)$ and $F''_j(z_j)$ from formulae (2.9) and (2.10), and then use (1.8) to obtain the final expressions for the components of the displacement and stresses in the half-plane.

## 5. EXAMPLE: THE DYNAMIC PROBLEM OF A PUNCH WHICH IS RIDIGLY ATTACHED TO AN ELASTIC HALF-PLANE

As an example of the use of the results obtained in the previous section, we will consider the following basic mixed DTE problem. In the semi-infinite interval $x \geqslant 0$ of the boundary of the elastic half-plane $y < 0$ at time $t = 0$, suppose that a punch, rigidly coupled to the half-plane, is set in motion. The law of motion of the punch is given by the equations $u = u_0(x, t)$, $v = v_0(x, t)$. Before the punch starts to move, the half-plane is at rest. Thus, the boundary and initial conditions of the problem are of the form (4.1), with $\sigma_0(x, t) = \tau_0(x, t) = 0$, and its general solution is given by formulae (1.8), (2.9), (2.10), (4.20) and (4.21) with $\Sigma_0(\xi) = \mathrm{T}_0(\xi) = 0$.

The major interest in the problem of a punch concerns the stresses underneath it, an analysis of which uncovers typical properties of this kind of problem. We will now discuss how these stresses are calculated.

The functions $\Sigma(\xi)$ and $\mathrm{T}(\xi)$ are RT of the values of the normal and shear stresses on the boundary of the half-plane. The stresses underneath the punch can be found at once from (4.21) using $\Sigma(\xi)$ and $\mathrm{T}(\xi)$ (without having to calculate the functions $F''_j(z_j)$), after applying to (4.21) the inverse RT (cf. (2.5))

$$\begin{Bmatrix} \sigma_y(x, 0, t) \\ \tau_{xy}(x, 0, t) \end{Bmatrix} = \mathrm{Re}\frac{1}{\pi i}\int_\Gamma \begin{Bmatrix} P_1^+(x, t, c) \\ P_2^+(x, t, c) \end{Bmatrix} dc \tag{5.1}$$

Taking (4.20) and the notation of (2.6) and (4.12) into account, for $x \geqslant 0$, in the complex plane of $c$, the integrands in (5.1) have branch points $c = \pm c_2, \pm c_1, x/t$ and, for $x/t < c_2$, are analytic in the entire $c$-plane with cuts $-\infty < \mathrm{Re}\, c - c_j$, $x/t < \mathrm{Re}\, c < \infty$, $\mathrm{Im}\, c = 0$ except possibly for simple poles of the functions $P_j^+(c)$.

The position of the contour $\Gamma$ was discussed in Section 1. By deforming along the cut $[x/t, \infty]$, we can transform the integrals in (5.1) into integrals over the interval $(x/t, \infty)$, traversed twice. Then, substituting (4.20) into (5.1) with $\Sigma_0 = \mathrm{T}_0 = 0$, using the notation of (1.6) we have

$$\begin{Bmatrix} \sigma_y(x,0,t) \\ \tau_{xy}(x,0,t) \end{Bmatrix} = \int_{x/t}^{\infty} \mathrm{Re} \sum_{n=1}^{2} \begin{Bmatrix} B_{3-n}(c) \\ -1 \end{Bmatrix} S_n(x,t,c)dc \tag{5.2}$$

$$S_n(x,t,c) = -\frac{(-1)^{n-1}}{\pi^2}\frac{\nu_n(c)}{B_2(c)-B_1(c)}\left\{\left(\frac{x-ct}{x}\right)^{\alpha_n(c)} \times\right.$$

$$\left.\times \int_{-ct}^{0}\left(\frac{\xi'}{\xi'+ct}\right)^{-\alpha_n(c)} \frac{U_0'(\xi',c)+A_n(c)V_0'(\xi',c)}{\xi'-x+ct}d\xi' + \pi i[U_0'(x-ct,c)+A_n(c)V_0'(x-ct,c)]\right\}$$

We now take typical subintervals $x/t < c_2$, $c_2 < x/t < c_1$ and $c_1 < x/t < \infty$ of the interval of integration in (5.2) for $x/t < c_2$.

For $x/t > c_1$ from (4.12) with $c > c_1$ we have

$$Q = -2\gamma_1^*\gamma_2^* - \gamma, \quad N_j = -\gamma_j^*(1-\gamma_2^2)$$

$$R = \gamma^2 + 4\gamma_1^*\gamma_2^* \quad (\gamma_j^* = (c^2/c_j^2 - 1)^{1/2})$$

and, therefore, $\bar{R} \equiv R$, $\bar{Q} \equiv Q$, $\bar{N}_j \equiv N_j$. Then, from the comment made just before (4.16), from (4.16), (4.18) and (4.20) we obtain: $B_1 = \nu_1 = 0$, $B_2 = \nu_2 = \infty$ and $\alpha_n = 0$. In this case, expressions (5.2) describe the stresses under the punch at points that are not reached by the disturbances from the edge $x = 0$. These are the stresses that there would be if conditions $u = u_0(x, t)$ and $v = v_0(x, t)$ held on the entire half-plane boundary.

For $c_2 < x/t < \infty$ from (4.12), (4.15), (4.18) and (4.20) for $c_2 < c < c_1$, we have

$$Q = 2i\gamma_1\gamma_2^* - \gamma, \quad N_1 = i\gamma_1(1-\gamma_2^2), \quad N_2 = -\gamma_2^*(1-\gamma_2^2)$$

$$R = \gamma^2 - 4i\gamma_1\gamma_2^*, \quad A_1 = 2\gamma_2^*/\gamma, \quad A_2 = -\gamma_2^* \tag{5.3}$$

$$B_1 = A_2, \quad B_2 = A_1, \quad \alpha_1 = 0$$

$$\alpha_2 = \frac{1}{\pi}(\mathrm{arctg}\frac{4\gamma_1\gamma_2^*}{\gamma^2} - \mathrm{arctg}\gamma_1\gamma_2^*)$$

In this case, expressions (5.2) describe the stresses under the punch with disturbances in the form of a cylindrical longitudinal wave and a forward transverse wave coming from the edge of the punch.

For $x/t < c_2$, from (4.12), (4.15), (4.18) and (4.20) for $0 < c < c_2$, we find

$$\begin{gathered}\bar{Q} \equiv Q = 2\gamma_1\gamma_2 - \gamma, \quad N_j = i\gamma_j(1-\gamma_2^2), \quad \bar{N}_j = -N_j \\ A_1 = i\sqrt{\gamma_2/\gamma_1}, \quad A_2 = -A_1, \quad B_1 = A_2, \quad B_2 = A_1 \\ \bar{R} \equiv R = \gamma^2 - 4\gamma_1\gamma_2, \quad \alpha_1 = -\frac{1}{2\pi i}\ln\frac{K_+}{K_-} \\ \alpha_2 = -\frac{1}{2\pi i}\ln\frac{K_-}{K_+}, \quad K_\pm = 2\gamma_1\gamma_2 - \gamma \pm \sqrt{\gamma_1\gamma_2}(1-\gamma_2^2)\end{gathered} \tag{5.4}$$

In the interval $0 < c < c_2$ the function $K_+(c)$ is positive for $0 < c < c_R$, where $c_R$ is the solution of the Rayleigh equation ($R(c_R) = 0$), and is negative for $c_R < c < c_2$, and the function $K_-(c)$ is negative in the entire interval $0 < c < c_2$. Hence, the functions $\alpha_1(c)$ and $\alpha_2(c)$ are pure imaginary in the portion $c_R < c < c_2$ of the interval $0 < c < c_2$, and complex ($\operatorname{Re}\alpha_n(c) = 1/2$) in $0 < c < c_R$. Thus in the interval $0 < c < c_2$ it is better to put $\alpha_1(c)$ and $\alpha_2(c)$ in the form

$$\alpha_1(c) = \frac{1}{2}H(c_R - c) + i\alpha(c), \quad \alpha_2(c) = \bar{\alpha}_1(c) \tag{5.5}$$

$$\alpha(c) = \frac{1}{2\pi}\ln\left|\frac{K_+(c)}{K_-(c)}\right|, \quad H(c_R - c) = \begin{cases}1 & (c < c_R)\\ 0 & (c > c_R)\end{cases}$$

Taking the limit as $x \to 0$, from (5.2), using (5.3)–(5.5), we obtain

$$\begin{Bmatrix}\sigma_y(x,0,t)\\ \tau_{xy}(x,0,t)\end{Bmatrix} = \int_0^{c_1} \operatorname{Re}\sum_{n=1}^{2} x^{\beta_n(c)} f_n(x,t,c)\begin{Bmatrix}B_{3-n}(c)\\ -1\end{Bmatrix} K_n(t,c)dc \quad (x \to +0) \tag{5.6}$$

$$f_n(x,t,c) = \cos\left[\operatorname{Im}(\alpha_n(c))\ln\frac{ct}{x}\right] + i\sin\left[\operatorname{Im}(\alpha_n(c))\ln\frac{ct}{x}\right] \tag{5.7}$$

$$K_n(t,c) = -\frac{(-1)^{n-1}}{\pi^2}\frac{\nu_n(c)}{B_2(c)-B_1(c)}(ct)^{-\beta_n(c)}\int_{-ct}^{0}(-\xi')^{-\alpha_n(c)}(\xi'+ct)^{\alpha_n(c)-1}\times$$
$$\times[U_0'(\xi',c) + A_n(c)V_0'(\xi',c)]d\xi'$$
$$\beta_n(c) = -\operatorname{Re}\alpha_n(c)$$

As in the case of the corresponding static problem (see [2], for example), it follows from (5.6), taking (5.5) into account, that the stresses at the point $x = 0$ have a singularity of type $x^{-1/2}$, superimposed by oscillations which are typical of this kind of problem, and thus the stresses change sign an infinite number of times as $x \to 0$. We can see from (5.7) that here, unlike the static case, the characteristic size of the region of oscillations depends not only on the elastic constants (which appear in $\alpha_n$), but also on time, and is therefore a non-stationary quantity.

## 6. SOME CONCLUDING REMARKS

The method described here succeeds in combining the basic ideas of two independent approaches to the solution of dynamic problems, dating back to Fourier and D'Alembert: the method of integral transforms, in which the solutions are represented in the form of a continuous superposition of plane waves of exponential form (Fourier and Laplace integrals), and the Smirnov–Sobolev method, in which the solution is sought in the form of arbitrary analytic functions. This combined approach, by means of which the solutions can be represented as an integral superposition of arbitrary analytic functions corresponding to plane waves of arbitrary form, has, in particular, obviated the constraints of self-similarity (in relation to the Smirnov–Sobolev method), whilst retaining the possibility of reducing the IBVP to boundary-value problems of the theory of analytic functions (in relation to the conventional version of the method of integral transforms).

To some extent, representations (1.7) and (1.8) are the most general and, obviously, the most suitable representations of the kinematic and dynamic characteristics of the motion of a homogeneous and isotropic elastic medium in plane deformation, since they can (in principle) be combined with the Radon transformation to reduce the IBVP of the DTE to the substantially simpler boundary-value problems of the theory of functions of a complex variable.

## REFERENCES

1. PORUCHIKOV, V. B., *Methods of the Dynamic Theory of Elasticity*. Nauka, Moscow, 1986.
2. MUSKHELISHVILI, N. I., *Some Basic Problems of the Mathematical Theory of Elasticity*. Nauka, Moscow, 1966.
3. PETRASHEN', G. I., Fundamentals of the mathematical theory of the propagation of elastic waves. In *Problems of the Dynamic Theory of the Propagation of Seismic Waves*., 1978, No. 18, Nauka, Leningrad.
4. GAKHOV F. D., *Boundary-value Problems*. Fizmatgiz, Moscow, 1958.
5. MUSKHELISHVILI, N. I., *Singular Integral Equations*. Fizmatgiz, Moscow, 1962.
6. LAVRENT'YEV, M. A. and SHABAT, B. V., *Methods of the Theory of Functions of a Complex Variable*. Nauka, Moscow, 1987.
7. GEL'FAND, I. M., GRAYEV, M. I. and VILENKIN, N. Ya., *Integral Geometry and Associated Problems of the Theory of Representations*. Fizmatgiz, Moscow, 1962.
8. HELGASON, S., *The Radon Transformation*. Mir, Moscow, 1983.
9. SLEPYAN, L. I., *Fracture Mechanics*., Sudostroyeniye, Leningrad, 1981.
10. PARTON, V. Z. and BORISKOVSKII, V. G., *Dynamic Fracture Mechanics*. Mashinostroyeniye, Moscow, 1985.
11. FLITMAN, L. M., The dynamic problem of a punch on an elastic half-plane. *Prikl. Mat. Mekh*., 1959, **23**, 697–705.
12. FLITMAN, L. M., On the motion of a rigid strip-mass lying on an elastic half-space and excited by a seismic wave. *Prikl. Mat. Mekh*., 1962, **26**, 1043–1058.
13. ZVOLINSKII, N. V., FLITMAN, L. M., KOSTROV, B. V., and AFANAS'YEV, V. A., Some problems of the diffraction of elastic waves. In *Application of the Theory of Functions in the Mechanics of a Continuous Medium*., 1965, Vol. 1, 432–443. Nauka, Moscow.
14. GALIN, L. A., *Contact Problems of the Theory of Elasticity*. Gostekhizdat, Moscow, 1953.

*Translated by* R.L.